JOURNAL OF COMPUTATIONAL PHYSICS 40, 183–201 (1981)

An Application of Network Theory to the Solution of Implicit Navier–Stokes Difference Equations*

R. AMIT,† C. A. HALL, AND T. A. PORSCHING

Institute for Computational Mathematics and Applications,
Department of Mathematics and Statistics,
University of Pittsburgh, Pittsburgh, Pennsylvania 15261

Received December 11, 1979; revised April 29, 1980

Many natural implicit discretizations of the Navier–Stokes equations can, with the proper identifications, be regarded as systems defining flows on associated networks. In this paper we describe a method which, through the introduction of a different set of network variables, significantly reduces the size of the original system. The method avoids the need to compute pressures, and produces velocities that are exactly discrete divergence free. We illustrate the technique by applying it to the implicit finite difference equations of Krzhivitski and Ladyzhenskaya and an implicit MAC-like finite difference system.

1. INTRODUCTION

In 1966 Krzhivitski and Ladyzhenskaya [3, 4] announced a finite difference scheme for the Navier–Stokes equations. As a Navier–Stokes discretization, it possesses many desirable theoretical properties. It is linear, even though the convection terms are included. Its difference equations possess a unique solution which is unconditionally stable in the discrete L_2 norm. Most importantly, it generates functions which converge to a weak solution [5] of the continuous problem as the discretization parameters tend to zero.

Yet with all of this to recommend it, the scheme does not seem to have been used to any extent as a computational tool. The reason for this is suggested by Chorin [1] who states that the scheme "... is not readily applied in practical calculation." This opinion is echoed by Jamet *et al.* [2] when they write that the scheme "... conduirait á résoudre á chaque pas dans le temps, un systéme complexe d'équation's linéaires." Apparently, the linear system generated by the difference scheme is too "complex" to be efficiently solved at each time step. As we shall see in the next section, the difference equations are natural discretizations of the differential equations, and so in a sense are no more complex than the original equations. Rather, the complexity results because all of the discrete primitive unknowns, pressures *and* velocities,

* This work supported by the Electric Power Research Institute under Contract RP-963 and by the Air Force Office of Scientific Research under Contract F49620-79-C-0100.

† Present address: 11 Ben Gurion Avenue, Tel Aviv, Israel.

0021-9991/81/030183-19$02.00/0

appear simultaneously in the system. In fact in two space dimensions, if there are N mesh points at which the solution is to be determined, then for large N one must solve $O(3N)$ non-trivially coupled equations at each time step. This should be compared with the $O(N)$ equations of the conventional explicit MAC method [6].

One of the aims of this paper is to show that by the use of elementary network theoretic properties of the finite difference mesh, the original system may be transformed into a *completely equivalent* reduced system involving only $O(N)$ equations and unknowns. The new unknowns are termed *dual variables* because of their relationship to the dual network of the finite difference grid. Once the dual variables are known, the original primitive variables may be recovered by simple back substitution. With this device then, the Krzhivitski–Ladyzhenskaya system may be regarded as equivalent in terms of size to the MAC system.

In the next section we present the Krzhivitski–Ladyzhenskaya equations and survey their properties. Section 3 contains the network theory and the transformation to the dual variable system. In Section 4 we show that this same idea may also be applied to a linearized implicit MAC-like system, and in Section 5 we generalize the method to treat inhomogeneous boundary conditions. Section 6 contains the details of a driven cavity computation using the dual variable MAC system of Section 4. The final section consists of additional remarks and observations on the method.

2. The Krzhivitski–Ladyzhenskaya (K–L) Equations

Let Ω be a bounded region of the x–y plane with boundary $\partial\Omega$ and let $[0, T]$ be a finite time interval. If $Q \equiv \Omega X[0, T]$, then the continuous problem is to find a velocity vector $\mathbf{V} \equiv (u, v)$: $Q \to R^2$, and a pressure p: $Q \to R$, which satisfy the Navier–Stokes equations,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mu \nabla^2 \mathbf{V} + \mathbf{F}, \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

and the boundary-initial conditions

$$\mathbf{V}\,|_{\partial\Omega} = 0, \tag{3}$$

$$\mathbf{V}\,|_{t=0} = \mathbf{V}^0. \tag{4}$$

Here for convenience we have assumed that the constant density is unity. Furthermore, μ is the viscosity, $\nabla =$ the gradient operator, $\mathbf{F} = (e, f)$ the (known) body force, and $\mathbf{V}^0$ an initial velocity field which vanishes on $\partial\Omega$ and satisfies (2).

To solve this problem we follow Krzhivitski and Ladyzhenskaya [3, 4] and overlay the x–y plane with a mesh of lines $x = kh$, $y = kh$, $k = 0, \pm 1, \pm 2, \ldots$, where $h > 0$ is the mesh spacing. Furthermore, we let $\Delta t = T/M$, M being a positive integer.

If w is any scalar (or vector) function defined on the mesh points at time levels $0, \Delta t, \ldots, M\,\Delta t$, we denote its value at $x = ih$, $y = jh$, $t = m\,\Delta t$ by w_{ij}^m. We also define the following mesh operators:

$$
\begin{aligned}
I w_{ij}^m &= w_{ij}^m, \\
S_x w_{ij}^m &= w_{i+1,j}^m, \\
\nabla_x w_{ij}^m &= w_{ij}^m - w_{i-1,j}^m, \\
\delta_x^2 w_{ij}^m &= w_{i+1,j}^m - 2w_{ij}^m + w_{i-1,j}^m.
\end{aligned}
$$

The reader will recognize these, respectively, as the identity, forward shift, backward difference, and second-centered-difference operators in the x direction. The corresponding y directional operators S_y, ∇_y and δ_y^2, as well as the backward time difference operator ∇_t, are analogously defined. Moreover, $\Delta_x = S_x \nabla_x$ and $\Delta_y = S_y \nabla_y$ are the forward difference operators, and $\nabla_h \equiv (\nabla_x, \nabla_y)$ is the discrete gradient.

With the aid of these operators we may write the K–L discretizations of (1)–(2) as follows:

$$
\begin{aligned}
&\frac{1}{\Delta t} \nabla_t \mathbf{V}_{ij}^m + \frac{1}{2h} (S_x + I)(u_{ij}^{m-1} \nabla_x \mathbf{V}_{ij}^m) + \frac{1}{2h} (S_y + I)(v_{ij}^{m-1} \nabla_y \mathbf{V}_{ij}^m) \\
&\quad = -\frac{1}{h} \nabla_h p_{ij}^m + \frac{\mu}{h^2} (\delta_x^2 + \delta_y^2) \mathbf{V}_{ij}^m + \mathbf{F}_{ij}^m, \qquad (5)
\end{aligned}
$$

$$
\frac{1}{h} (\Delta_x u_{ij}^m + \Delta_y v_{ij}^m) = 0, \qquad m = 1, \ldots, M, \qquad (6)
$$

where of course $\mathbf{V}_{ij}^m \equiv (u_{ij}^m, v_{ij}^m)$, $\mathbf{F}_{ij}^m = (e_{ij}^m, f_{ij}^m)$.

The spatial domains of definition of the equations are important. To describe them we introduce the (closed) mesh squares $\sigma_{ij} = [ih, (i+1)h] \times [jh, (j+1)h]$, $i, j = 0, \pm 1, \ldots$, and define $\bar{\Omega}_h \equiv \bigcup \sigma_{ij}$, where the union is over all $\sigma_{ij} \in \Omega \cup \partial\Omega$. Then (5) holds at the interior mesh points of $\bar{\Omega}_h$ and (6) holds at those points of $\bar{\Omega}_h$, where a pressure p_{ij}^m is introduced by virtue of (5). In view of (3), we also set $\mathbf{V}_{ij}^m = \mathbf{0}$ on $\partial\bar{\Omega}_h$, the boundary of $\bar{\Omega}_h$. We assume that $\bar{\Omega}_h$ is *mesh point connected* in the sense that each pair of distinct mesh points in $\bar{\Omega}_h$ where (6) holds may be connected by a path of mesh square sides lying in the *interior* of $\bar{\Omega}_h$.

There are now exactly as many equations as unknowns. However, Eqs. (6) are not independent since they sum identically to zero. Therefore, the additional equation

$$
\sum_{i,j} p_{ij}^m = 0, \qquad (7)
$$

is appended to the system.

In Fig. 1 we have illustrated a simple region in which $\bar{\Omega}_h$ consists of 11 mesh squares and is mesh point connected. Equation (5) holds at the mesh points marked

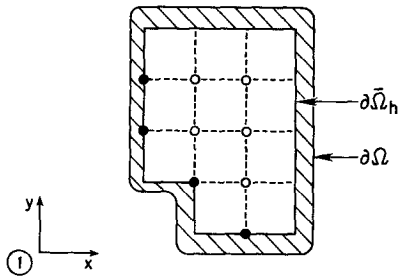

FIG. 1. The domain of (5) and (6).

○, while (6) holds at those points plus the points marked ●. Thus there are five discrete momentum equations (5), eight independent discrete continuity equations (6) and Eq. (7). Because of the boundary conditions, there are correspondingly five unknown velocity vectors and nine unknown pressures (i.e., 19 unknown scalars).

The unique solvability of the K–L equations, as well as their unconditional stability follows from an equality involving the discrete L_2 norm $\|\cdot\|_h$ induced by the inner product,

$$(\mathbf{U}, \mathbf{V})_h \equiv h^2 \sum_{i,j} \mathbf{U}_{ij} \cdot \mathbf{V}_{ij},$$

where $\mathbf{U}$ and $\mathbf{V}$ are any 2-dimensional vector functions defined on the mesh points of $\bar{\Omega}_h$. Utilizing certain discrete analogues of integral identities involving a solution of (1)–(4), Krzhivitski and Ladyzhenskaya derive the following equality for any velocity vector $\mathbf{V}$ satisfying (5) and (6),

$$\|\mathbf{V}^m\|_h^2 - \|\mathbf{V}^{m-1}\|_h^2 + \|\mathbf{V}^m - \mathbf{V}^{m-1}\|_h^2 + \frac{2\mu\,\Delta t}{h^2}\left(\|\Delta_x \mathbf{V}^m\|_h^2 + \|\Delta_y \mathbf{V}^m\|_h^2\right)$$

$$= 2\,\Delta t(\mathbf{F}^m, \mathbf{V}^m)_h, \qquad m = 1,..., M. \tag{8}$$

If (5) and (6) constitute a homogeneous system, then $\mathbf{F}^m = \mathbf{V}^{m-1} = \mathbf{0}$, and it follows immediately from (8) that $\mathbf{V}^m = \mathbf{0}$. But then (5) implies that all the p_{ij}^m are equal, and by (7) this common value must be zero. Therefore, by the alternative principle for linear systems, *the K–L equations have a unique solution.* Furthermore, (8) yields

$$\|\mathbf{V}^m\|_h \leqslant \|\mathbf{V}^{m-1}\|_h + 2\,\Delta t\,\|\mathbf{F}^m\|_h,$$

and from this we have

$$\|\mathbf{V}^m\|_h \leqslant \|\mathbf{V}^0\|_h + 2\sum_{k=1}^{m} \|\mathbf{F}^k\|_h\,\Delta t$$

$$< \|\mathbf{V}^0\| + 2\int_0^T \|\mathbf{F}\|\,dt + o(1),$$

where $\|\cdot\|$ is the L_2 norm on Ω. This last inequality is a statement of the *unconditional stability* of the K–L equations.

The proof that the approximations generated by the K–L equations converge to the weak solution of (1)–(4) involves detailed function theoretic arguments and is too lengthy to outline here. We remark, however, that the keystone of the proof is again the equality (8). See [4].

It is clear that if $\bar{\Omega}_h$ contains N mesh points, then the size of the K–L system is $O(3N)$. Moreover, the subsystems of (5) and (6) do not uncouple in any way. Contrasted to this, the usual MAC equations [6] require the solution of a discrete Poisson equation for the pressures, and its size is only $O(N)$. In the next section, we show how to transform the K–L equations into an equivalent system whose size is also only $O(N)$.

3. The K–L Dual Variable System

Given the K–L equations of the previous section, we can define an associated directed network $\mathcal{N}$ as follows. The *nodes* of $\mathcal{N}$ are isomorphic to the mesh points (ih, jh) of $\bar{\Omega}_h$ where (6) holds. The directed *links* of $\mathcal{N}$ are isomorphic to certain ordered pairs of network nodes $((i-1)h, jh) \to (ih, jh)$ or $(ih, (j-1)h) \to (ih, jh)$. Link$((i-1)h, jh) \to (ih, jh)$, which is directed *away* from node $((i-1)h, jh)$ and *toward* node (ih, jh), is in $\mathcal{N}$ precisely if u_{ij}^m is *not* determined by the boundary conditions. Similarly, link$(ih, (j-1)h) \to (ih, jh)$ is in $\mathcal{N}$ if the boundary conditions do not determine v_{ij}^m. Since we have assumed that $\bar{\Omega}_h$ is mesh point connected, the network $\mathcal{N}$ is connected.

With $\mathcal{N}$ defined in this manner we say that the *state* at node (ih, jh) is p_{ij}^m, that the *flow* on link$((i-1)h, jh) \to (ih, jh)$ is u_{ij}^m, and that the flow on link$(ih, (j-1)h) \to (ih, jh)$ is v_{ij}^m. All of this is succinctly illustrated in the molecule shown in Fig. 2.

Now suppose that $\mathcal{N}$ contains N nodes and L links. Then we can put its nodes and links into 1–1 correspondences with the first N and L positive integers, respectively. This induces a relabeling of the states and flows, say $\{p_{ij}^m\} \leftrightarrow \{p_1^m, \ldots, p_N^m\}$ and

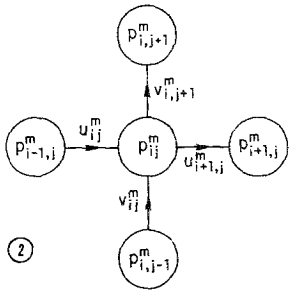

FIG. 2. Defining molecule of K–L network.

$\{u_{ij}^m, v_{ij}^m\} \leftrightarrow \{w_1^m, \ldots, w_L^m\}$. The body forces are then also relabeled $\{e_{ij}^m, f_{ij}^m\} \leftrightarrow \{g_1^m, \ldots, g_L^m\}$. Let $A = [a_{nl}]$ be the $N \times L$ node-link incidence matrix of $\mathcal{N}$. That is,

$$\begin{aligned} a_{nl} &\equiv +1 \quad \text{if link } l \text{ is directed away from node } n, \\ &\equiv -1 \quad \text{if link } l \text{ is directed toward node } n, \\ &\equiv 0 \quad \text{otherwise.} \end{aligned}$$

It is a standard result of graph theory [7], that A has rank $N-1$. Furthermore, from its definition it follows that (6) may be written as

$$A\mathbf{W}^m = \mathbf{0}, \tag{9}$$

where $\mathbf{W}^m = (w_1^m, \ldots, w_L^m)^T$.

The L-dimensional linear system (5) may also be written in vector form as

$$\frac{1}{\Delta t}(\mathbf{W}^m - \mathbf{W}^{m-1}) + B_{m-1}\mathbf{W}^m = \frac{1}{h}A^T\mathbf{P}^m + \mathbf{g}^m, \tag{10}$$

where $\mathbf{P}^m = (p_1^m, \ldots, p_N^m)^T$, $\mathbf{g}^m = (g_1^m, \ldots, g_L^m)^T$, and B_{m-1} is the $L \times L$ matrix which results from the combined convection and viscous stress discretizations. Each row of B_{m-1} contains no more than five non-zero entries. The general form of the diagonal entry is

$$\frac{4\mu}{h^2} - \frac{1}{2h}[\Delta_x u_{ij}^{m-1} + \Delta_y v_{ij}^{m-1}] = \frac{4\mu}{h^2},$$

while the off-diagonal entries are all of the form

$$-\frac{\mu}{h^2} \pm \frac{1}{2h} w_k^{m-1},$$

where w_k^{m-1} is one of the velocities u_{ij}^{m-1}, v_{ij}^{m-1}, $u_{i+1,j}^{m-1}$ or $v_{i,j+1}^{m-1}$, the minus sign being used with either of the first two velocities and the plus sign with the last two.

Now let C be a fundamental matrix of $\mathcal{N}$. That is, C is a $L \times (L-N+1)$ matrix whose columns are linearly independent and are null vectors of A. Matrix C then has the following properties:

(i) $AC = 0$, i.e., C is orthogonal to A,

(ii) If $A\mathbf{Z} = 0$, then $\mathbf{Z} = C\boldsymbol{\gamma}$ for some $\boldsymbol{\gamma} \in R^{L-N+1}$, i.e., the columns of C form a basis for the null space of A.

With this in mind we rewrite (10) as

$$Q_{m-1}\mathbf{W}^m = \Delta t\, A^T\mathbf{P}^m + \mathbf{b}^m, \tag{11}$$

where

$$Q_{m-1} \equiv h(I + \Delta t\, B_{m-1})$$

and

$$\mathbf{b}^m \equiv h(\mathbf{W}^{m-1} + \Delta t\, \mathbf{g}^m)$$

contain only known information. Multiplying (11) by C^T and using the orthogonality of A and C, we obtain

$$C^T Q_{m-1} \mathbf{W}^m = C^T \mathbf{b}^m.$$

But by (9) and property (ii) of C,

$$\mathbf{W}^m = C\boldsymbol{\gamma}^m, \tag{12}$$

for some $\boldsymbol{\gamma}^m \in R^{L-N+1}$. Thus,

$$C^T Q_{m-1} C\boldsymbol{\gamma}^m = C^T \mathbf{b}^m. \tag{13}$$

Since $L = O(2N)$, the size of the system (13) is $O(N)$. Once $\boldsymbol{\gamma}^m$ is known, the velocities are easily recovered from (12). If desired, the pressure differences $A^T \mathbb{P}^m$ may be obtained from (11). These quantities, along with (7) then determine the pressures. Note, however, that the pressures *are not needed* to advance the velocities from time level to time level.

Note further that (6) is satisfied when $\mathbf{W}^m$ is constructed from (12) using *any* vector $\boldsymbol{\gamma}^m$. It follows that the velocities obtained from (12) are *exactly discrete divergence free* in the sense that round-off errors in the solution of (13) do not affect the degree to which $\mathbf{W}^m$ satisfies (6).

As we shall see below, matrix C may be constructed so that there are at most two non-zero entries in any row. Moreover, these entries will be either $+1$ or -1. With this choice of C, the only accuracy limitation in (12) is that associated with one floating point subtraction.

The unique solvability of (13) follows from that of the K–L equations and the following lemma.

LEMMA. *Every solution of* (9) *and* (11) *determines a solution of* (13). *Conversely, every solution of* (13) *determines a solution of* (9) *and* (11).

Proof. The first part follows from the construction of (13). Conversely, if $\boldsymbol{\gamma}^m$ solves (13) and $\mathbf{W}^m \equiv C\boldsymbol{\gamma}^m$, then $A\mathbf{W}^m = \mathbf{0}$ and

$$C^T[Q_{m-1}\mathbf{W}^m - \mathbf{b}^m] = \mathbf{0}.$$

Hence $Q_{m-1}\mathbf{W}^m - \mathbf{b}^m$ is in the null space of C^T. But the dimension of this null space is $L - (L - N + 1) = N - 1$. Since the rank of A^T is $N - 1$ and since $C^T A^T =$

$(AC)^T = 0$, the columns of A^T span the null space of C^T. Therefore, there is a vector $\Delta t\, \mathbf{P}^m \in R^N$ such that

$$Q_{m-1}\mathbf{W}^m - \mathbf{b}^m = A^T(\Delta t\, \mathbf{P}^m). \qquad \text{Q.E.D.}$$

The remaining question concerning the reduced system (13) is the construction of the fundamental matrix C. But this is easily accomplished using another well-known graph theoretic result involving the *elementary cycles* of $\mathcal{N}$. These are the simple closed paths in $\mathcal{N}$ which do not surround any other closed paths in $\mathcal{N}$. Loosely speaking they are the boundaries of the holes in $\mathcal{N}$. If $C = [c_{lk}]$, then we generate the kth column of C by walking around the kth elementary cycle of $\mathcal{N}$ in (say) the counterclockwise direction (the manner in which the cycles of $\mathcal{N}$ are numbered is irrelevant). Then

$$\begin{aligned} c_{lk} &\equiv +1 \quad \text{if link } l \text{ is directed away from the last node we have passed through,} \\ &\equiv -1 \quad \text{if link } l \text{ is directed toward the last node we have passed through,} \\ &\equiv 0 \quad \text{otherwise.} \end{aligned}$$

It is well known [7] that this procedure will generate exactly $L - N + 1$ independent columns and that the resulting C is orthogonal to A.

It is not difficult to see that if C is constructed in the above manner, then the rows of $C^T Q_{m-1} C$ generally contain at most 13 non-zero entries, and the structure of the non-zero couplings resembles that of the discrete biharmonic operator. This is illustrated in Fig. 3 and shows the unknowns that appear in the equation for γ_c^m. This figure also shows the elementary cycles corresponding to the γ's. Note that the arrow heads point at the nodes towards which the link is directed.

Since $\mathcal{N}$ is a planar network, it has a dual, $\mathcal{N}^*$ [7]. The γ's may then be associated with the states of $\mathcal{N}^*$ and, consequently, we call them *dual variables.*

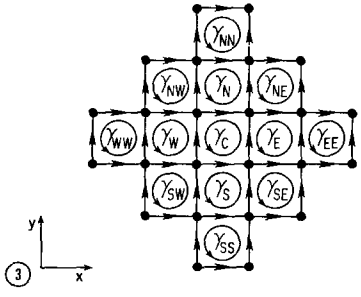

FIG. 3. The 13 point coupling of Eq. (13).

Figure 4 gives the network $\mathcal{N}$ constructed from the region $\bar{\Omega}_h$ of Fig. 1. With the node, link and cycle labelings of this figure, the incidence and fundamental matrices A and C^T are those given below,

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C^T = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

One verifies directly that $AC = 0$. Furthermore, since $L = 10$, $N = 9$, the number of unknown primitive variables (i.e., pressures and velocities) is $L + N = 19$ as previously noted. However, the number of dual variables is $L - N + 1 = 2$.

4. Another Application

It should be clear from the previous section that a dual variable system exists for any system of Navier–Stokes difference equations having a directed network $\mathcal{N}$ on which:

(i) the discrete continuity equations may be interpreted as Kirchhoff node laws;

(ii) the discrete momentum equations may be interpreted as generalized Ohm's laws relating the link flows to pressure drops across the links.

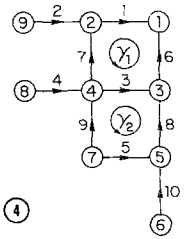

FIG. 4. The K–L network of Fig. 1.

As another application of this general observation, we consider a linearized implicit MAC-type discretization of (1)–(4). The *explicit* version of these equations was studied in [8–9]. Utilizing the notation of the previous sections, we have

$$\frac{1}{\Delta t}\nabla_t u^m_{i-1/2,j} + \frac{u^{m-1}_{i-1/2,j}}{h}\left[\theta(u^{m-1}_{i-1/2,j})\,\nabla_x u^m_{i-1/2,j} + (1-\theta(u^{m-1}_{i-1/2,j}))\,\nabla_x u^m_{i+1/2,j}\right]$$

$$+\frac{\bar{v}^{m-1}_{i-1/2,j}}{h}\left[\theta(\bar{v}^{m-1}_{i-1/2,j})\,\nabla_y u^m_{i-1/2,j} + (1-\theta(v^{m-1}_{i-1/2,j}))\,\nabla_y u^m_{i-1/2,j+1}\right]$$

$$= -\frac{1}{h}\nabla_x p^m_{i,j} + \frac{\mu}{h^2}(\delta_x^2+\delta_y^2)\,u^m_{i-1/2,j} + e^m_{i-1/2,j}, \tag{14}$$

$$\frac{1}{\Delta t}\nabla_t v^m_{i,j-1/2} + \frac{\bar{u}^{m-1}_{i,j-1/2}}{h}\left[\theta(\bar{u}^{m-1}_{i,j-1/2})\,\nabla_x v^m_{i,j-1/2} + (1-\theta(\bar{u}^{m-1}_{i,j-1/2}))\,\nabla_x v^m_{i+1,j-1/2}\right]$$

$$+\frac{v^{m-1}_{i,j-1/2}}{h}\left[\theta(v^{m-1}_{i,j-1/2})\,\nabla_y v^m_{i,j-1/2} + (1-\theta(v^{m-1}_{i,j-1/2}))\,\nabla_y v^m_{i,j+1/2}\right]$$

$$= -\frac{1}{h}\nabla_y p^m_{i,j} + \frac{\mu}{h^2}(\delta_x^2+\delta_y^2)\,v^m_{i,j-1/2} + f^m_{i,j-1/2}, \tag{15}$$

$$\frac{1}{h}(\Delta_x u^m_{i-1/2,j} + \Delta_y v^m_{i,j-1/2}) = 0, \tag{16}$$

where

$$\theta(Z) = 0 \quad \text{if} \quad Z \leqslant 0$$

$$= 1 \quad \text{if} \quad Z > 0,$$

$$\bar{v}^{m-1}_{i-1/2,j} = \frac{(I+S_x)}{2}\frac{(I+S_y)}{2}\,v^{m-1}_{i-1,j-1/2},$$

$$\bar{u}^{m-1}_{i,j-1/2} = \frac{(I+S_x)}{2}\frac{(I+S_y)}{2}\,u^{m-1}_{i-1/2,j-1}.$$

Equations (14) and (15) hold, respectively, on the sets $\{((i-1/2)h, jh) \in \bar{\Omega}^0_h\}$, and $\{(ih, (j-1/2)h) \in \bar{\Omega}^0_h\}$, where $\bar{\Omega}^0_h = \bar{\Omega}_h - \partial\bar{\Omega}_h$. Equation (16) holds at points where a pressure p^m_{ij} occurs because of (14) or (15). Near $\partial\bar{\Omega}_h$ (14)–(16) introduce velocity approximations at points on and exterior to $\partial\bar{\Omega}_h$. In accordance with (3), these are set to zero.

As before, a normalizing equation such as (7) is required to define the pressure field uniquely.

Note that the convection terms are discretized by means of "upwind differences," and that the pressures, x-directional, and y-directional velocities are defined on the familiar interlaced grids of the MAC method.

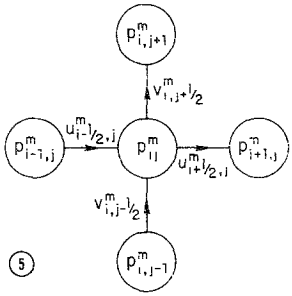

FIG. 5. Defining molecule of MAC network.

For these equations the directed network $\mathcal{N}$ has as its nodes the mesh points corresponding to the points where (16) holds. The links of $\mathcal{N}$ are given by the sets

$$\{((i-1)h, jh) \to (ih, jh) \mid ((i-1/2)h, jh) \in \bar{\Omega}_h^0\},$$

and

$$\{(ih, (j-1)h) \to (ih, jh) \mid (ih, (j-1/2)h) \in \bar{\Omega}_h^0\}.$$

The flows and states of $\mathcal{N}$ are then defined in an obvious manner. Figure 5 illustrates the molecule for this network.

Equations (14)–(16) may now be recast in the form (9) and (11) and the dual variable system follows as before.

With regard to the solvability of this system, of course the lemma of the previous section remains valid. However, there is no equality analogous to (8) from which to establish its unique solvability via that of (14)–(16). By examining the dual variable system directly, it can be shown as in [11] that for fixed h a unique solution exists at each time level provided that Δt is sufficiently small. In practice this sufficient condition appears to be unnecessarily restrictive.

In Fig. 6 we illustrate a labeled MAC network for the region $\bar{\Omega}_h$ of Fig. 1. This should be compared with the earlier K–L network of Fig. 4. Since for the MAC

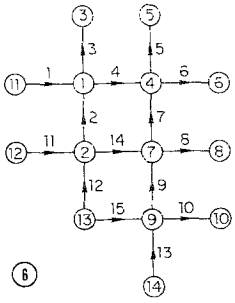

FIG. 6. The MAC network of Fig. 1.

network $L = 15$, $N = 14$, the number of unknowns in (14)–(16) is now 29. However, the MAC dual variable system contains only two unknowns as before.

5. Inhomogeneous Boundary Conditions

Most practical problems in computational fluid dynamics involve inhomogeneous boundary conditions instead of those given by (3). For example, (3) may be replaced by

$$\mathbf{V}|_{\partial\Omega} = \mathbf{V}^*, \tag{17}$$

where $\mathbf{V}^* = (u^*, v^*)$ is a specified non-zero velocity distribution on $\partial\Omega$ satisfying (in view of (2))

$$\oint_{\partial\Omega} u^*\, dy - v^*\, dx = 0.$$

While it is true that this more general problem can in theory be reduced to one with homogeneous boundary conditions by constructing a divergence free vector field which satisfies (17) (see [10]), the actual construction may be tedious, and it seems desirable to treat (17) directly in the difference equations.

Moreover, it is sometimes necessary to deal with *mixed* boundary conditions of the form

$$\mathbf{V}|_{\partial\Omega_1} = \mathbf{V}^*, \tag{18a}$$

$$p|_{\partial\Omega_2} = p^*, \tag{18b}$$

where p^* is a prescribed pressure distribution on $\partial\Omega_2$, $\Omega_1 \cap \Omega_2 = \varnothing$, and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. For a reasonable discretization of the problem in terms of the primitive variables, it is usually not difficult to account for (18). For example, if the difference equations are defined on an approximating region such as $\bar{\Omega}_h$, then one can translate the boundary conditions to $\partial\bar{\Omega}_h$ in some obvious way, say by projection. The translation of condition (18a) to $\partial\bar{\Omega}_h$ defines a set $\partial\bar{\Omega}_{h1} \subset \partial\bar{\Omega}_h$ on which the translated condition holds, modifies the source term in the discrete momentum equations, and introduces a (possible) *non-zero* source into the discrete continuity equations. On the other hand, when (18b) is translated to $\partial\bar{\Omega}_h$, it defines an analogous set $\partial\bar{\Omega}_{h2} \subset \partial\bar{\Omega}_{h1}$, again modifies the source in the discrete momentum equations, but *eliminates* a continuity equation for every discrete pressure it specifies. Note that if there is any such pressure, then (7) is no longer required. Indeed, its retention overspecifies the problem.

Assuming that it is possible to construct a connected network $\mathcal{N}_*$ on which the difference equations may be interpreted as Kirchhoff node laws and Ohm's laws, the

incorporation of the effects of (18) will, because of the above remarks, generally lead to the following revised forms of (9) and (11),

$$A_* \mathbf{W}^m = \mathbf{s}^m, \tag{19}$$

$$Q_{m-1} \mathbf{W}^m = \Delta t\, A_*^T \mathbf{P}^m + \mathbf{b}_*^m, \tag{20}$$

where A_* and $\mathbf{b}_*$ are a modified incidence matrix and source vector, and $\mathbf{s}^m$ is the (non-zero) source introduced by the boundary velocities.

We can not directly apply the method of Section 3 to produce the dual variable system for (19) and (20). There are two reasons for this. First, if $\mathbf{s}^m \neq \mathbf{0}$, then the vector $\mathbf{W}^m$ no longer lies in the null space of A_*. Second, if $\mathcal{N}_*$ has N nodes and L links, of which, say $L_0 > 0$, are incident on points of $\partial\bar{\Omega}_{h2}$, then A_* has full row rank. Its null space is therefore of dimension $L - N$, and the columns of the fundamental matrix constructed from the elementary cycles of $\mathcal{N}_*$ in the manner of Section 3 will not in general span it. In fact, since these cycles will be confined to a connected subnetwork of $\mathcal{N}_*$ consisting of N nodes and $L - L_0$ links, they will provide only $(L - L_0) - N + 1 = (L - N) - (L_0 - 1)$ basis vectors, leaving a deficiency of $L_0 - 1$ such vectors. Nevertheless, the procedure of Section 3 can be modified to provide a dual variable system in this case.

The first step in the generation of this system is to obtain a *particular solution* of the underdetermined system (19). Such a solution is easily determined by using a *spanning tree* of $\mathcal{N}_*$, and algorithms for the determination of a spanning tree are well known [7]. With the spanning tree available, one sets the velocities on the links of $\mathcal{N}_*$ which are not in the tree equal to zero. Beginning with its outermost extremities, one then proceeds through the nodes of the tree. As each node is encountered, one link velocity is determined from the continuity equation at that node. Certain continuity equations may contain extra degrees of freedom (i.e., velocities) due to the presence of the specified pressures. These velocities may also be set to zero without violating (19).

Use of the spanning tree in effect permutes (19) so that it is in upper triangular form. Its solution is then obtained by a back substitution process. An example of this technique is given below.

Now let $\mathbf{Y}^m$ be the particular solution of (19) obtained above. Defining the vector

$$\mathbf{Z}^m = \mathbf{W}^m - \mathbf{Y}^m, \tag{21}$$

we have from (19)

$$A_* \mathbf{Z}^m = \mathbf{0}, \tag{22}$$

and from (20)

$$Q_{m-1} \mathbf{Z}^m = \Delta t\, A_*^T \mathbf{P}^m + \mathbf{d}^m, \tag{23}$$

where

$$\mathbf{d}^m \equiv \mathbf{b}_*^m - Q_{m-1}\mathbf{Y}^m$$

is a vector of known information. Equations (22) and (23) are again of the form (9) and (11). Thus, if C_* is a fundamental matrix for A_*, we find that the dual variable system is

$$C^T Q_{m-1} C_* \gamma^m = C^T \mathbf{d}^m, \tag{24}$$

The velocities are then given by

$$\mathbf{W}^m = C_* \gamma^m + \mathbf{Y}^m. \tag{25}$$

It remains to construct the matrix C_*. As before, we can obtain $(L-N)-(L_0-1)$ of its columns from the elementary cycles of $\mathcal{N}_*$ by the method of Section 3. The remaining $L_0 - 1$ columns may be generated by the use of *pseudo-cycles*. These are connected paths through $\mathcal{N}_*$ that begin and end with distinct links that are incident on points of $\partial\bar{\Omega}_{h2}$. One walks from one end of each pseudo-cycle to the other, and uses the prescription given in Section 3 to generate the corresponding column of C_*. If the links incident on points of $\partial\bar{\Omega}_{h2}$ are ordered so that their link numbers are $l(1),\ldots, l(L_0)$, then for $k = 1,\ldots, L_0 - 1$, there is a pseudo-cycle, say the kth, that begins on link $l(k)$ and ends on link $l(k+1)$. The $L_0 - 1$ columns that these pseudo-cycles generate are independent since the kth pseudo-cycles contains link $l(k+1)$, and this is neither in any elementary cycle of $\mathcal{N}_*$, nor in pseudo-cycles $1,\ldots, k-1$.

To illustrate these ideas, suppose that velocity and pressure distributions are given on the boundary of the region of Fig. 1 in such a way that their translation to $\partial\bar{\Omega}_h$ defines $\partial\bar{\Omega}_{h1}$ as the most northern and eastern segments of $\partial\bar{\Omega}_h$, and $\partial\bar{\Omega}_{h2}$ as the remainder of $\partial\bar{\Omega}_h$. If we adopt the difference equations of Section 4, then the induced network $\mathcal{N}_*$ is shown in Fig. 7. Note that this differs from the network of Fig. 6 through the omission of nodes 11 through 14. These nodes are not included in $\mathcal{N}_*$

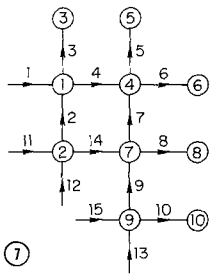

FIG. 7. Modified MAC network.

since there is no discrete continuity equation at them. For $\mathcal{M}_*$ we have $N = 10$, $L = 15$, $L_0 = 5$, and the incidence matrix is

$$A_* = \left[\begin{array}{cccccccccc|ccccc} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A spanning tree for $\mathcal{M}_*$ may be obtained by discarding link 14. Extra degrees of freedom are then given by the velocities on links 11, 12, 13, and 15. When the velocities on these links are set to zero, (19) may be solved as an upper triangular system. In fact, $\mathcal{M}_*$ has been labeled so that the submatrix of A_* consisting of all its rows and first 10 columns is upper triangular (i.e., The permutation induced by this particular tree is the identity).

Since there is only $(L - N) - (L_0 - 1) = 1$ elementary cycle in $\mathcal{M}_*$, $L_0 - 1 = 4$ pseudo-cycles are required to complete the construction of C_*. One such set is given by the paths corresponding to the link sets $\{11, 2, 1\}$, $\{12, 11\}$, $\{15, 9, 14, 12\}$ and $\{13, 15\}$. The resulting C_*^T is

$$C_*^T = \left[\begin{array}{cccccccccc|ccccc} 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right].$$

Again one verifies directly that $A_* C_* = 0$. It follows that there are five unknown dual variables for this example.

6. A Driven Cavity Calculation

To evaluate the effectiveness of the dual variable approach, a computer program, DUVAL, was written which utilizes the equivalent reduced MAC equations (14)–(16) of Section 4. In DUVAL, these equations are solved directly at each time level by a "frontal method" as described in [11, 14]. Inhomogeneous boundary conditions are treated in the manner of Section 5.

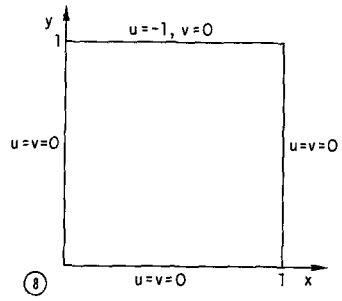

FIG. 8. The driven cavity.

As a sample calculation, we consider the numerical solution of the familiar driven cavity problem [12, 13]. The boundary conditions for this problem are shown in Fig. 8. In (1) we set $\mu = 0.025$, which is equivalent to the case of a Reynolds' number of 400.

For the numerical solution the cavity was divided into 400 uniform mesh squares ($h = 0.05$) so that system (14)–(16) contains 1160 equations and unknowns. In contrast, the size of the equivalent dual variable system is $L - N + 1 = 760 - 400 + 1 = 361$. Since we are primarily interested in the steady state solution, a uniform time step of 20 was used. The calculation started from the null initial condition $\mathbf{V}^0 \equiv \mathbf{0}$, and remained completely stable.

After 16 time steps, $\max[|\nabla_t u_{i-1/2,j}|, |\nabla_t v_{i,j-1/2}|] < 1.5 \times 10^{-5}$. Figure 9 shows the computed velocity direction field and Fig. 10 the u component of the centerline velocity at this time. Also shown in this latter figure is a curve of the same quantity obtained by Burggraf [15]. Although there is general qualitative agreement between the two numerical solutions, the DUVAL solution shows considerably more dissipation than Burggraf's. This is due to the well known numerical diffusion effect introduced by the use of upwind differences for the convection terms. As pointed out by Tuann and Olson [13] and others, this effect is particularly severe for the driven

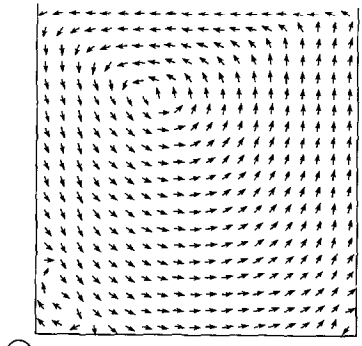

FIG. 9. Velocity direction field after sixteen time steps.

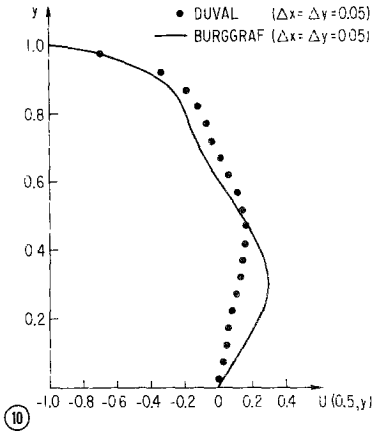

FIG. 10. Centerline velocity.

cavity problem, and we do not suggest that in this case the DUVAL solution is highly accurate. Furthermore, we emphasize that its inaccuracies must be laid at the feet of the original difference equations (14)–(16) and not the dual variable system, since the latter is generated by—and equivalent to—the former. If desired, the upwind differences could be replaced by the more accurate (but less stable) centered differences (see Section 7).

Computation times for this problem were ~59 CPU seconds per time step on a DEC-1099 computer and ~2.5 seconds per time step on a CDC-7600. The code was not optimized on either machine.

7. SUPPLEMENTARY REMARKS

In keeping with the network theme of this paper, we have identified the γ's with the states of the dual network $\mathcal{N}^*$. However, if Ω is simply connected, it is particularly easy to relate them to the values of a discrete stream function. For example, for the K–L Equations, we may put the columns of C, as generated in Section 3, into a natural 1–1 correspondence with a subset of the points $\{(i-1/2, j-1/2)h\}$. If the components of γ in (12) are labeled accordingly, then one verifies that (12) is equivalent to the equations

$$u_{ij} = \gamma_{i-1/2,j+1/2} - \gamma_{i-1/2,j-1/2},$$
$$v_{ij} = \gamma_{i-1/2,j-1/2} - \gamma_{i+1/2,j-1/2}.$$

Thus, if we define a mesh function ψ such that

$$\psi_{i-1/2,j-1/2} = h\gamma_{i-1/2,j-1/2} \qquad \text{if} \quad \gamma_{i-1/2.j-1/2} \text{ is defined by (12)}$$
$$= 0 \qquad \text{otherwise,}$$

then

$$u_{ij} = \frac{1}{h} \Delta_y \psi_{i-1/2,j-1/2},$$

$$v_{ij} = -\frac{1}{h} \Delta_x \psi_{i-1/2,j-1/2},$$

so that ψ is a discrete stream function. A similar result holds for the MAC system of Section 4. Note, however, that a discretization of the well-known fourth order equation for the continuous stream function [13] will not result in the dual variable system (13). The implementation of such a discretization requires the setting of appropriate boundary conditions for the stream function, which may be the main disadvantage of this approach. By comparison, no auxilary conditions are imposed on the γ's, since all boundary conditions are applied to the primitive variables in the untransformed system.

Each directed network $\mathcal{N}$ uniquely defines its associated incidence matrix A. However, *any* basis for the null space of A will generate a corresponding fundamental matrix C. The procedure given in Section 3 generates a particularly sparse C. Moreover, it lends itself to automation and has been included as part of the preprocessing stage of the computer program DUVAL, which accepts as input a *user oriented* description of the flow region.

The theory of dual variable reduction directly generalizes to the case of three space dimensions. The primitive and reduced systems contain $O(4N)$ and $O(2N)$ unknowns, respectively. One can still use the notion of a cycle to generate columns of C (see [7]). However, since $\mathcal{N}$ is no longer planar, the definition of an elementary cycle as we have given it does not apply.

Since the subject of this paper is the presentation and development of the dual variable method, we have purposely avoided the issue of the solution of the reduced system (13). Obviously, many of the variants of Gaussian elimination may be used for a *direct* solution of (13). The frontal method, mentioned in conjunction with DUVAL, is such a variant. We found it appealing to use this technique since it conserves storage by alternating the solution of system (13) with its generation, and this latter task is conveniently associated with a sequential processing of the links of $\mathcal{N}$. However, we remark that since $\lim_{\Delta t \to 0} C^T Q_m C = hC^T C$, a positive definite, symmetric matrix, (13) should also be amenable to solution by *iterative* methods such as SOR (at least for Δt sufficiently small).

Finally, we emphasize that the dual variable method is independent of the type of finite differences used to discretize the convection and viscous stress terms in the momentum equations. These influence the entries of Q and $\mathbf{b}$ in system (13) but not its size. In particular, centered differences could have been employed for the convection terms in the MAC system of Section 4 without degrading the variable reduction capability of the dual variable method.

REFERENCES

1. A. J. CHORIN, *Math. Comp.* **23** (1969), 341.
2. P. JAMET, P. LASCAUX, AND P.-A. RAVIART, *Numer. Math.* **16** (1970), 93.
3. A. KRZHIVITSKI AND O. A. LADYZHENSKAYA, *Soviet Phys. Dokl.* **11** (1966), 212 (Translation).
4. A. KRZHIVITSKI AND O. A. LADYZHENSKAYA, *in* "Boundary Value Problems of Mathematical Physics, IV," Amer. Math. Soc. Transl., Providence, R.I., 1968; *Proc. Math. Inst. Steklov* **92** (1966), 105.
5. E. HOPF, *Math. Nachr.* **4** (1951), 213.
6. F. H. HARLOW AND F. E. WELCH, *Phys. Fluids* **8** (1965), 2182.
7. C. BERGE AND A. GHOUILA-HOURI, "Programming, Games and Transportation Networks," Methuen, London, 1965.
8. T. A. PORSCHING, *Nucl. Sci. Eng.* **64** (1977), 177.
9. T. A. PORSCHING, *Numer. Math.* **29** (1978), 291.
10. O. A. LADYZHENSKAYA, "The Mathematical Theory of Viscous Incompressible Flow," Gordon & Breach, New York, 1969.
11. R. AMIT, C. CULLEN, C. HALL, G. MESINA, AND T. PORSCHING, *in* "Proceedings Third International Conference on Finite Elements in Flow Problems, Banff, Alberta, Canada, 1980.
12. NASA Langley Staff, "Numerical Studies of Incompressible Viscous Flow in a Driven Cavity," Report NASA SP-378, Natural Technical Information Service, Springfield, VA, 1975.
13. S. TUANN AND M. D. OLSON, *J. Comput. Phys.* **29** (1978), 1.
14. B. IRONS, *Int. J. Numer. Meth. Eng.* **2** (1970), 5.
15. O. R. BURGGRAF, *J. Fluid Mech.* **24** (1966), 113.